Lecture 5/5/2015

LTI Systems have an impulse response $\delta(n) \to h(n)$

$\delta[n] \to h[n]$

"[ ]" = Discrete Time

"( )" = Continuous Time

Time Invariance: $\delta(n-k) \to h(n-k)$

$\delta[n-k] \to h[n-k]$

Linearity: $x(k)\delta(n-k) \to x(k)h(n-k)$

$x[k]\,\delta[n-k] \to x[k]\,h[n-k]$

Superposition: $\underbrace{\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)}_{x(n)} \to \underbrace{\sum_{k=-\infty}^{\infty} x(k)h(n-k)}_{y(n)}$

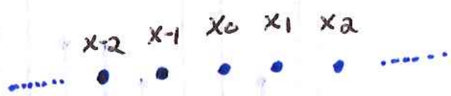
$$\therefore\ y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \Longrightarrow \sum_{k=-\infty}^{\infty} x(n-k)h(k) \Rightarrow x(n) * h(n)$$

convolution!

$h(n) * x(n) = x(n) * h(n)$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

$x(k)$



$\Rightarrow$

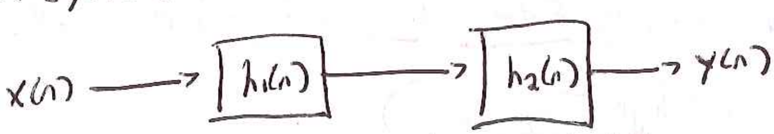
$h(k)$

$h(-k)$ $y(0)$

$h(1-k)$ $y(1)$

$h(-1-k)$ $y(-1)$

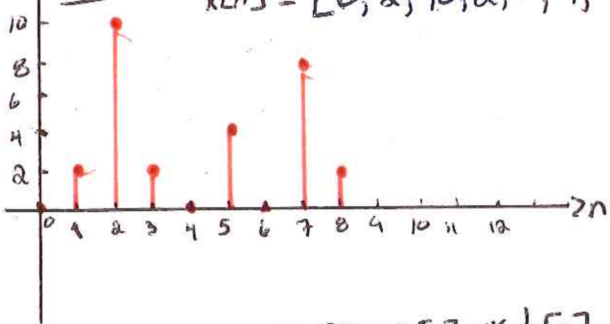
Cascaded systems:



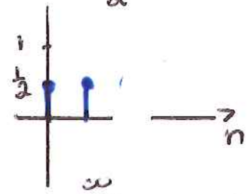
$y(n) = x(n) * h_1(n) * h_2(n)$

Example

$x[n] = [0, 2, 10, 2, 0, 4, 0, 8, 2]$



$h[n] = \frac{1}{2}[1, 1]$



$y[n] = x[n] * h[n] \rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$

$n = 0 \Rightarrow \sum_{k=-\infty}^{\infty} x[-k]\, h[k] = y[0] = 0$

$= x_0 h_0 = 0 \cdot \frac{1}{2} = 0$

$n = 1 \Rightarrow \sum_{k=-\infty}^{\infty} x[1-k]\, h[k] = y[1] = 1$

$= x_1 h_0 + x_0 h_1 = 2 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = 1$

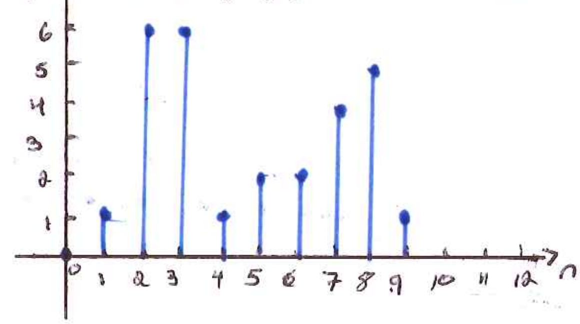
$n = 2 \Rightarrow \sum_{k=-\infty}^{\infty} x[2-k]\, h[k] = y[2] = 6$

$x_2 h_0 + x_1 h_1$

$10 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = 6$

If you work it all the way out, you get

$y[n] = [0, 1, 6, 6, 1, 2, 2, 4, 5, 1]$



$h[n] = \frac{1}{2}[1, 1]$ is a smoothing, averaging filter (low-pass filter) that attenuates or suppresses sharp changes (high frequencies)

Fourier Transform "FT"

Continuous Time Fourier Transform "CTFT"

For a signal $x(t)$, it's FT is : $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

The Inverse FT "IFT" of $X(j\omega)$: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$

Let's find the FT of a cosine wave with $x(t) = \cos(2\pi\omega_0 t)$

$$\therefore X(j\omega) = \int_{-\infty}^{\infty} \cos(2\pi\omega_0 t) e^{-j\omega t} dt$$

Use euler's identity $\cos(2\pi\omega_0 t) = \frac{e^{2\pi\omega_0 j t} + e^{-2\pi\omega_0 j t}}{2}$

$$\therefore X(j\omega) = \int_{-\infty}^{\infty} \left[\frac{e^{j2\pi\omega_0 t} + e^{-j2\pi\omega_0 t}}{2} \right] \times e^{-j\omega t} dt$$

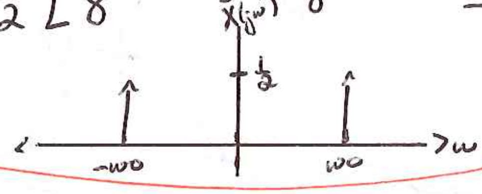
$$= \frac{1}{2} \int_{-\infty}^{\infty} \left[e^{-2\pi j(\omega-\omega_0)t} + e^{-2\pi j(\omega+\omega_0)t} \right] dt$$

By definition

$$e^{j\omega_0 t} \xrightarrow{FT} 2\pi\delta(\omega-\omega_0)$$

a delta with magnitude 2π at ω_0

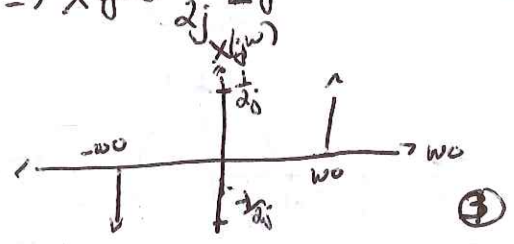
$$= \frac{1}{2} \left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0) \right]$$



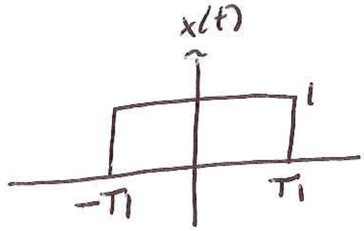
Similar for sine ⇒ $\sin(2\pi\omega_0 t) = \frac{e^{2\pi\omega_0 j t} - e^{-2\pi\omega_0 j t}}{2j}$

$$\therefore X(j\omega) = \int_{-\infty}^{\infty} \frac{1}{2j} \left[e^{j2\pi\omega_0 t} - e^{-j2\pi\omega_0 t} \right] e^{-j\omega t} dt$$

$\rightarrow X(j\omega) = \frac{1}{2j} \left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0) \right]$



$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



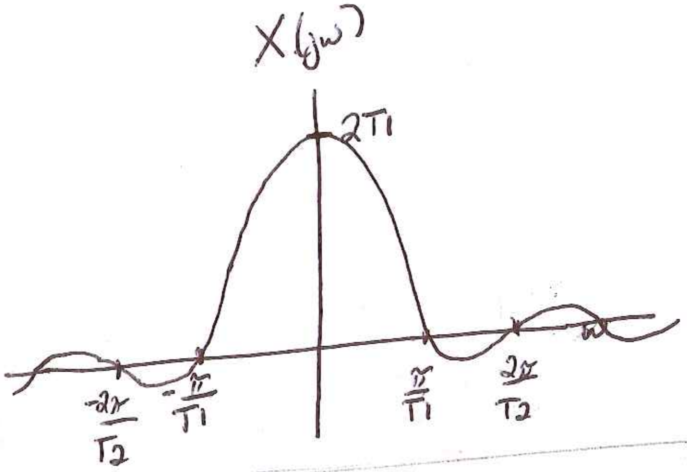
$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\, dt = \frac{1}{-j\omega}\left[e^{-j\omega t}\right]_{-T_1}^{T_1}$$

$$= \frac{1}{-j\omega}\left[e^{-j\omega T_1} - e^{j\omega T_1}\right] \quad \text{Distribute negative} \Rightarrow \frac{1}{j\omega}\left[e^{j\omega T_1} - e^{-j\omega T_1}\right]$$

Manipulate with 2 + j

$$\frac{2}{\omega}\left[\frac{e^{j\omega T_1} - e^{-j\omega T_1}}{2}\right] = \sin(\omega T_1)$$

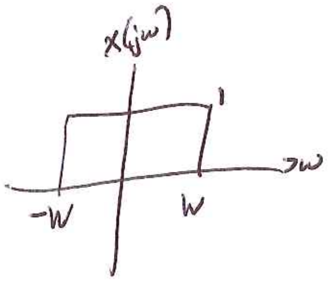
$$\therefore X(j\omega) = 2\frac{\sin(\omega T_1)}{\omega}$$

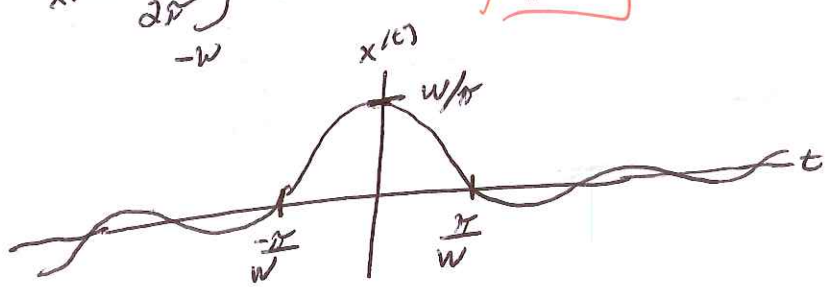


Let's Try an Inverse!

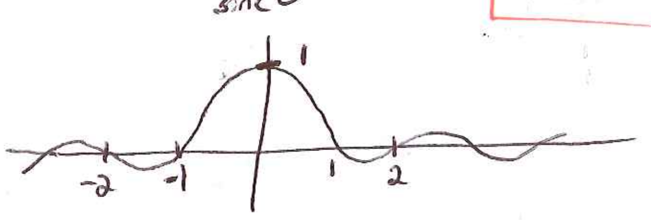
$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\, d\omega = \boxed{\frac{\sin Wt}{\pi t}}$$





sinc function : $\text{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$



Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

FT Properties

Convolution $y(t) = h(t) * x(t) \xleftrightarrow{\mathcal{F}} Y(j\omega) = H(j\omega) X(j\omega)$

linearity $ax(t) + by(t) \xleftrightarrow{\mathcal{F}} aX(j\omega) + bY(j\omega)$

time Shifting $x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(j\omega)$

Frequency Shifting $e^{j\omega_0 t} x(t) \xleftrightarrow{\mathcal{F}} X(j(\omega-\omega_0))$

Multiplication $x(t)y(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) Y(j(\omega-\theta)) \, d\theta$

Basic FT Pairs

$$e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi \delta(\omega-\omega_0)$$

$$\cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} \pi [\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

$$\sin(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{\pi}{j} [\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]$$

$$x(t) = 1 \xleftrightarrow{\mathcal{F}} 2\pi \delta(\omega)$$

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0 & |t| > T_1 \end{cases} \xleftrightarrow{\mathcal{F}} \frac{2 \sin \omega T_1}{\omega}$$

$$\frac{\sin Wt}{\pi t} \xleftrightarrow{\mathcal{F}} X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1$$

$$\delta(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0}$$

Time and Frequency Domain Relationship.

CTFT : $X(j\omega)$ → magnitude and phase representation : $X(j\omega) = |X(j\omega)| e^{j\angle X(j\omega)}$

DTFT : $X(e^{j\omega})$ " : $X(e^{j\omega}) = |X(e^{j\omega})| e^{j\angle X(e^{j\omega})}$

Frequency Response of LTI Systems

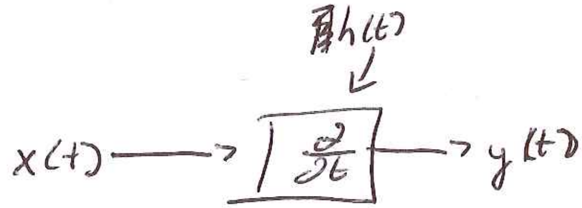
CTFT: $Y(j\omega) = H(j\omega) X(j\omega)$

DTFT: $Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$

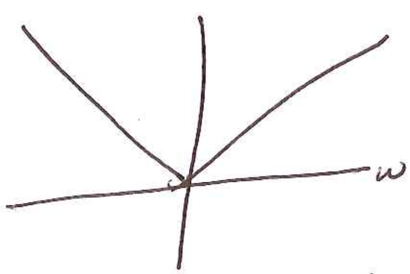
Filters

Differentiator

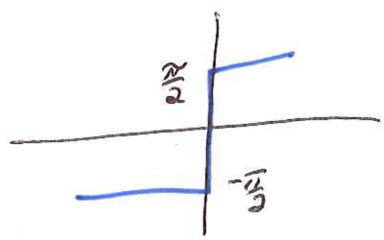
$$y(t) = \frac{d}{dt} x(t)$$



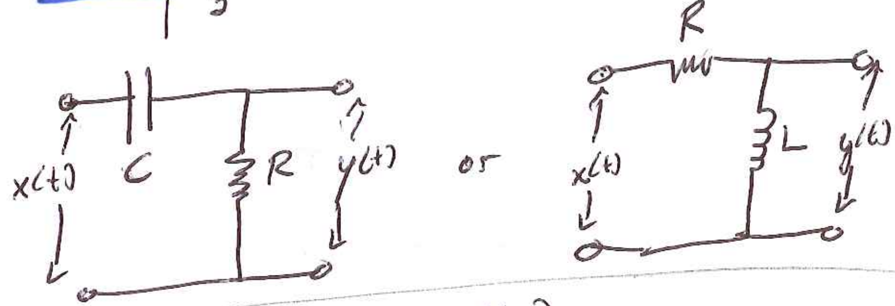
$|H(j\omega)|$



$\angle H(j\omega)$

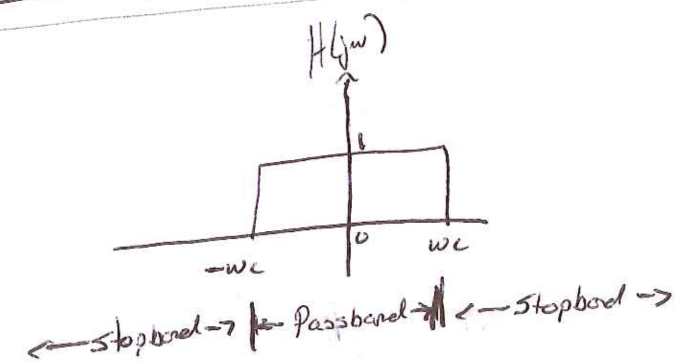


Analog : →

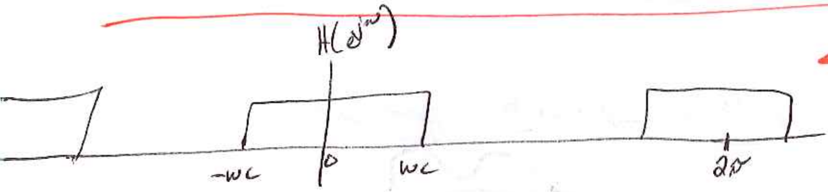


Frequency Selective Filters

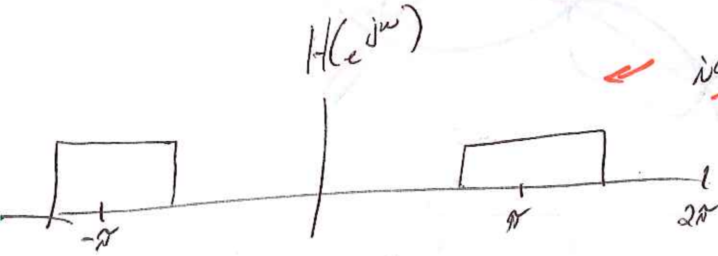
$$H(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$



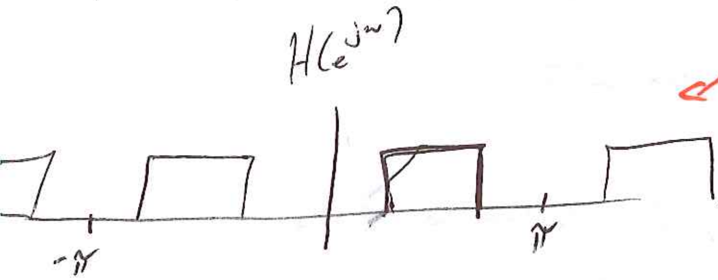
Discrete time Frequency Selection Filters



← ideal Lowpass

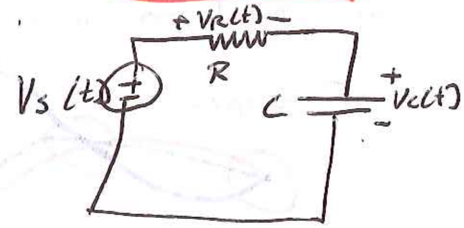


← ideal Highpass



← ideal Bandpass

RC LPF



$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

input $v_s(t) = e^{j\omega t}$

$\therefore$ output $= H(j\omega)e^{j\omega t}$

We get $RC\frac{d}{dt}\left[H(j\omega)e^{j\omega t}\right] + H(j\omega)e^{j\omega t} = e^{j\omega t}$

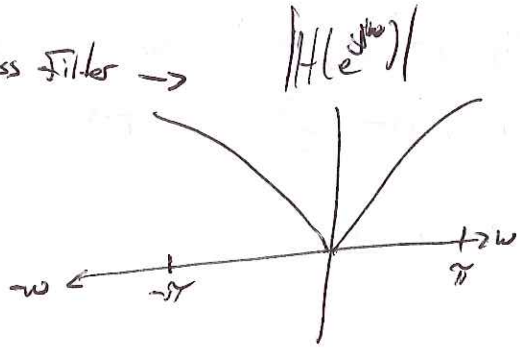
Rearrange $H(j\omega)e^{j\omega t} = \frac{1}{1+RCj\omega}e^{j\omega t}$

or $H(j\omega) = \frac{1}{1+RCj\omega}$

$\therefore h(t) = \frac{1}{RC}e^{-t/RC}u(t)$

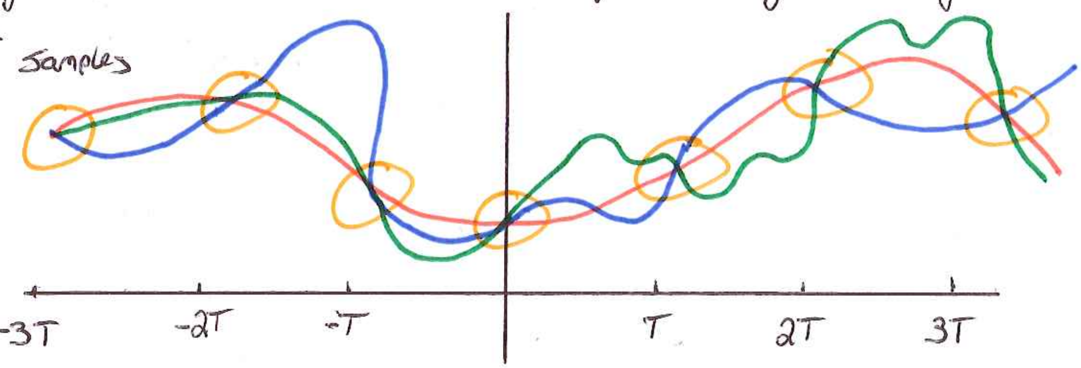
Filters are usually **NOT** ideal!

simple highpass filter →



Sampling Theorem as it relates to the frequency domain

In general, an infinite number of signals can generate a given set of samples



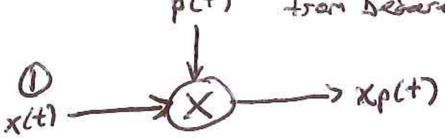
Three CT signals with identical values at integer multiples of T

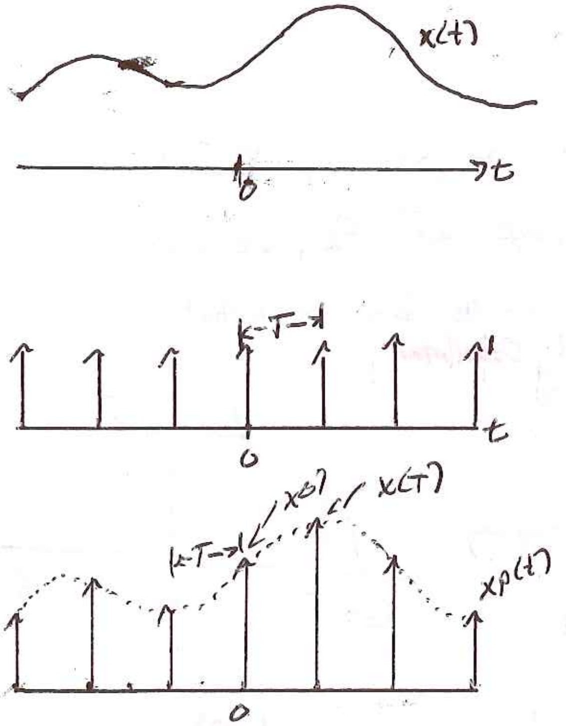
However, we will see that if a signal is band-limited — i.e. if it's Fourier Transform is zero outside a finite band of frequencies — and if the samples are taken sufficiently close together in relation to the highest frequency present in the signal, then the samples can uniquely specify the signal, and we can reconstruct it perfectly!

Impulse Train Sampling

Signal $x(t)$ is sampled by a periodic pulse train $p(t)$ "sampling function" with period T as the sampling period $p(t)$, $\omega_s = 2\pi/T$ as the sampling frequency

$\therefore x_p(t) = x(t)\,p(t)$ where $p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$ and as we know from before $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$





$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT)$$

By the multiplication Property

$$X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)\,P(j(\omega-\theta))\,d\theta$$

also, we know the FT of a pulse train

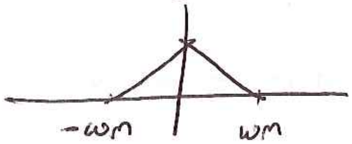
$$P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

and since convolution with an impulse just shifts a signal [i.e. $X(j\omega) * \delta(\omega-\omega_0) = X(j(\omega-\omega_0))$]

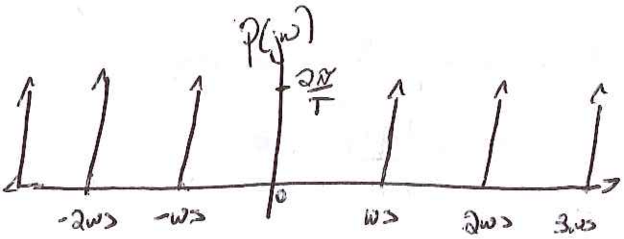
we have $$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

That is, $X_p(j\omega)$ is a periodic function of ω consisting of a superposition of shifted replicas of $X(j\omega)$, scaled by $\frac{1}{T}$

if $X(j\omega)$ → where ω_m is maximum frequency content and we have

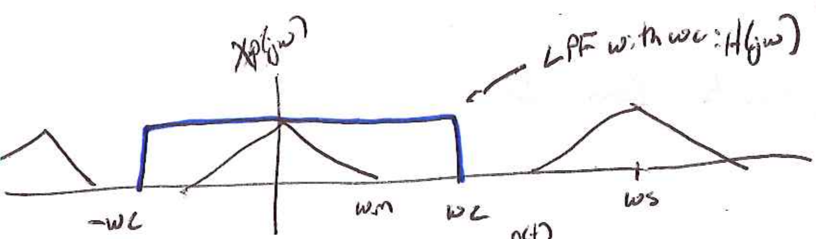


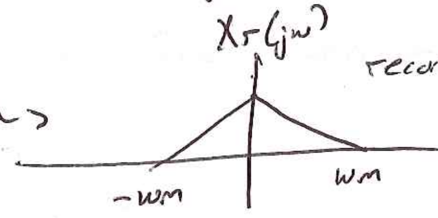
$\omega_m < (\omega_s - \omega_m)$ or equivalently, $\omega_s > 2\omega_m$, we have no overlap



if $\omega_s < 2\omega_m$, there is overlap.

Thus, $\omega_s > 2\omega_m$, $x(t)$ can be exactly recovered from $x_p(t)$ by means of lowpass filtering





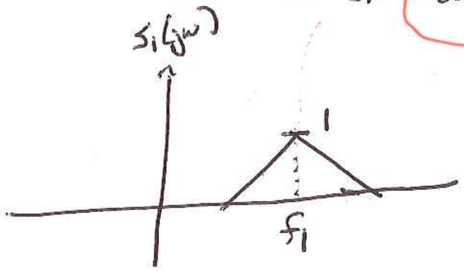
undersampling results in aliasing

Frequency Translation

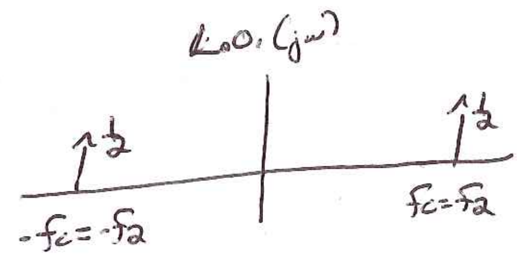
have a signal $s_1(t)$ with center Frequency at f_1, I want to move that signal to have its center at f_2, we will translate the Frequency or modulate the wave to s_2 such that

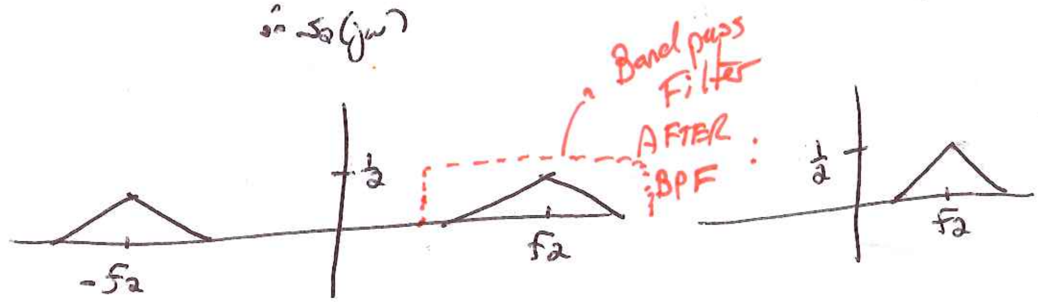
$s_2 = s_1(t)\cos(2\pi f_c t)$ where $f_c = f_2$

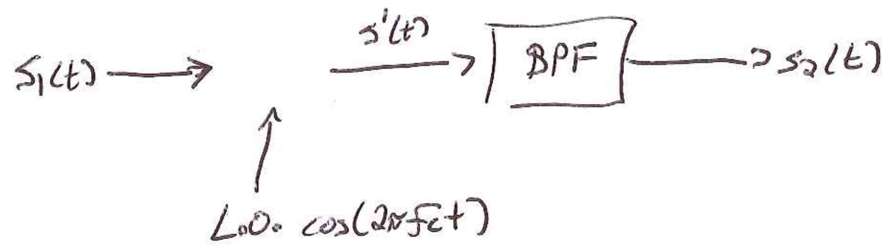
→ L.O. Local Oscillator



Multiplication in time is Convolution in Frequency





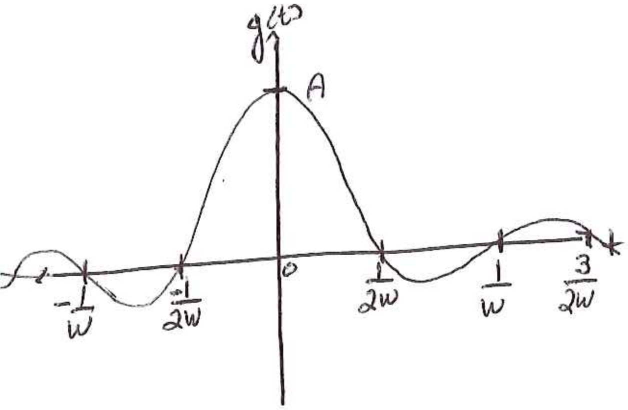


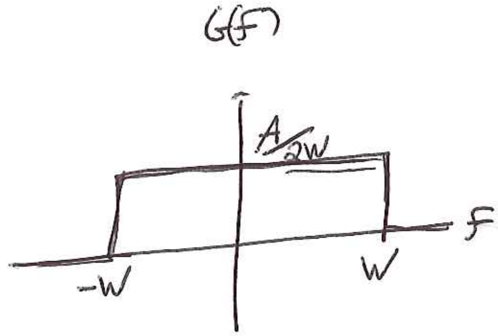
works in a similar way for translating down in Frequency.

This is how we take low, baseband signals + upconvert them to higher frequencies for other applications like WiFi, Radio, Fiber, Satellite communications, etc.

Source converts up↑, Sink converts down ↓

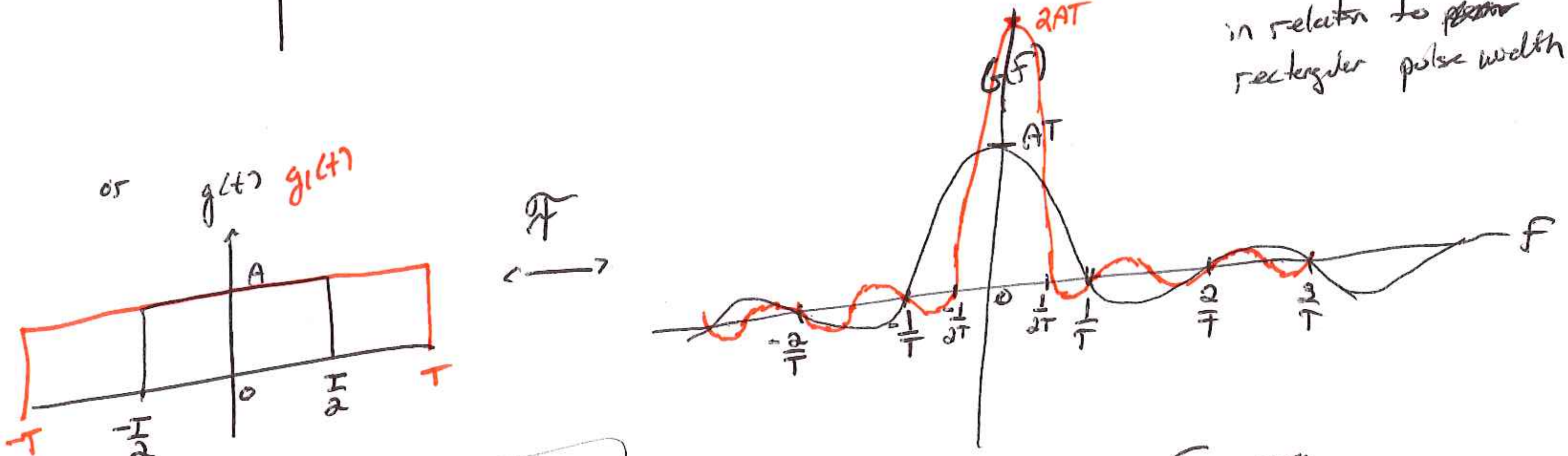
sinc in time: $g(t) = A\,\text{sinc}(2Wt)$





First zero crossing in relation to rectangular pulse width

or

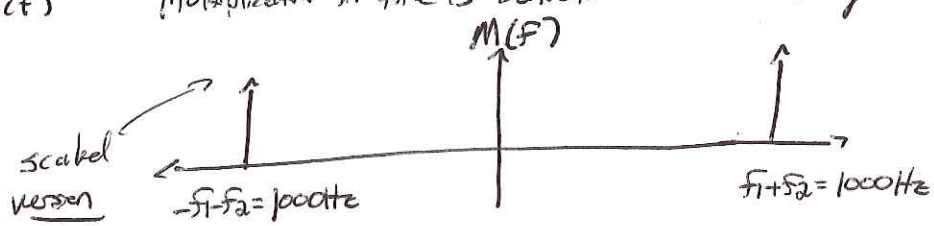


wider in time is narrower in frequency
wider in frequency is narrower in time

$m(t) = s_1(t) \cdot s_2(t)$

$s_1(t) = \cos(2\pi 400t)$
$s_2(t) = \cos(2\pi 600t)$

Multiplication in time is convolution in frequency



Filters in relation to Decibels. Logarithmic unit for comparing ratio of physical values/quantity usually on intensity or power ratio.

Generally need a reference level. Let's say whispers are 30 decibels. + a lawn mower is 90 decibels, 60dB more: really means $10^{60/10} \to 10^6$ times more as loud! 1 million times louder!